ELSEVIER

Discrete Mathematics 130 (1994) 171–176

DISCRETE MATHEMATICS

The edge-forwarding index of orbital regular graphs

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Received 13 February 1988

Abstract

We define a graph as *orbital regular* if there is a subgroup of its automorphism group that acts regularly on the set of edges of the graph as well as on all its orbits of ordered pairs of distinct vertices of the graph. For these graphs there is an explicit formula for the *edge-forwarding index*, an important traffic parameter for routing in interconnection networks. Using the arithmetic properties of finite fields we construct infinite families of graphs with low edge-forwarding properties. In particular, the edge-forwarding index of Paley graphs is determined. A connection with the Waring problem over finite fields and the coset weight enumeration of certain cyclic codes is established.

1. Introduction

The edge-forwarding index of a graph was introduced in [8] in the following way.

Let Γ be a finite simple undirected connected graph of order n. A *routing* R of Γ is a set of $n(n-1)$ elementary paths, such that each ordered pair of distinct vertices of Γ is linked by one of them.

The *load* of an edge e in a routing R is the number of paths of R through it, and is denoted by $\Pi(\Gamma, R, e)$. The *edge-forwarding index of a routing* R in a graph Γ is by definition

$$\Pi(\Gamma, R) = \max_{e \in E(\Gamma)} \Pi(\Gamma, R, e)$$

and the *edge-forwarding index of a graph* Γ is

$$\Pi(\Gamma) = \min_{R} \Pi(\Gamma, R),$$

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SSDI 0012-365X(92)00528-3

where the min is over all possible routings in Γ. The physical interpretation is immediate: $\Pi(\Gamma)$ is a traffic parameter on the edges of Γ, and we want to find graphs Γ with Π as small as possible. We shall also use the *merit factor* f_Γ defined for graphs with n nodes and maximum valency Δ as

$$f_\Gamma = \Pi(\Gamma)\Delta/n\log_{\Delta-1} n.$$

This parameter is most useful for asymptotic considerations because of the lower bound of [8]

$$f_\Gamma \geqslant 2 + \mathrm{o}(1),$$

valid for fixed Δ and large n.

In this note we shall develop a symmetry argument which is analogous [8, Theorem 3.6] on the vertex-forwarding index of Cayley graphs. We shall obtain asymptotic bounds on Π for (constructive) infinite families of graphs. In particular, we determine exactly the index of the Paley graph Γ_q for every q. The connection with the Waring problem for finite fields and the coset enumeration of certain cyclic codes is pointed out.

2. Orbital regular graphs

We recall that a group G is said to *act regularly* on a finite set X if both the action is transitive and $|G| = |X|$. We shall denote by $(X)_2$ the set of pairs of *distinct* elements of the set X. Let $\Gamma = (V, E)$ be a connected graph and G its automorphism group. We shall say that Γ is *orbital regular* if

(i) there exists a subgroup H of G which acts regularly on each of its orbits on $(V)_2$,
(ii) one of these orbits is exactly E.

Such orbits are sometimes termed "orbitals" [1]. To compute the edge-forwarding index of an orbital regular graph we shall need the following Lemma 2.1 (Theorem 3.2 of [8]):

Lemma 2.1. *If there exists a routing of shortest paths in Γ for which the load of all edges is the same, then*

$$\Pi(\Gamma) = \frac{1}{|E|} \sum_{(u,v)\in V^2} d(u, v).$$

We are now in a position to state and prove the following theorem, which is our main motivation for introducing the class of orbital regular graphs.

Theorem 2.2. *In an orbital regular graph with n nodes the edge-forwarding index is*

$$\Pi(\Gamma) = \left(\frac{1}{|E|}\right) \sum_{(u,v)\in V^2} d(u, v).$$

Proof. To use Lemma 2.1, we need to show that there is a routing R of shortest paths which loads every edge equitably. We construct the routing R as follows.

In each orbit of G on $(V)_2$ we select an arbitrary ordered pair of vertices (x, y) and a pair of shortest paths between x and y. For every other pair of vertices in the same orbit the routing is conventionally defined as the image of the routing between x and y. There is no possible ambiguity in this definition since H acts regularly on the orbit. Moreover, since graph automorphisms preserve the distance, this routing is a routing of shortest paths.

Let $e \in E$ be an edge of Γ and $e' \neq e$ another edge. Then by conditions (i) and (ii) the edges of Γ are a single orbit of G on $(V)_2$, and there exists a unique automorphism ϕ in H which maps e onto $e' = \phi(e)$. If γ is a path of R through e, then there exists a unique γ' through e' with $\gamma' \in R$, namely $\gamma' = \phi(\gamma)$. □

3. Constructions

3.1. The Paley graph

Let q be a power of a prime $q \equiv 1 \pmod 4$. We consider the Cayley graph P_q with vertex set $\boldsymbol{F}_q$, the finite field with q elements, and additive generating set S the nonzero squares (also called quadratic residues) in $\boldsymbol{F}_q$. By the arithmetic condition $q \equiv 1$ mod 4 we have that $S = -S$, and, consequently, the graph is undirected. We obtain a graph with q vertices, degree $(q-1)/2$, and diameter 2 [3]. Since S is a subgroup of index 2 of the multiplicative group of $\boldsymbol{F}_q$, it has exactly one other coset besides itself in $\boldsymbol{F}_q$. The orbitals are the translates of these 2 cosets. Hence P_q is orbital regular. Knowing that there are exactly $(q-1)/2$ points at distance 2 from a given point, we obtain, from Lemma 2.1

$$\Pi(P_q) = (1/|E|)q((q-1)/2 + 2(q-1)/2),$$

and since $2|E| = q(q-1)/2$, we eventually obtain:

$$\Pi(P_q) = 6.$$

3.2. The Waring problem over a finite field

More generally we can consider the Cayley graph $\Gamma_{t,q}$ with associated group $(\boldsymbol{F}_q, +)$ and generating set $S_t = \{x^t \mid x \in \boldsymbol{F}_q - \{0\}\}$, with t a divisor of $q-1$. If $(q-1)/t$ is even, then $S_t = -S_t$ and the graph is undirected. Of course, this precaution is useless if q is a power of 2. In the cases we shall consider it is known that this graph is connected. A criterion for this is that $(q-1)/t$ does not divide $q'-1$, for some q' strictly dividing q. Again, S_t is a multiplicative subgroup of $\boldsymbol{F}_q$ of index t. Hence we have t nontrivial orbitals containing pairs of the form $\{0, x\}$, $x \neq 0$ on which S_t acts regularly by multiplication. In all, we get tq orbitals on which the semi-direct product $(\boldsymbol{F}_q, +)\cdot(S_t, \cdot)$ acts regularly.

The average distance in this graph does not seem to be known in general, so we shall use the following crude bound:

Lemma 3.1. *Let $\Gamma=(V, E)$ be an orbital regular graph, with n vertices, constant degree Δ, and diameter D. Then the merit factor is bounded above*

$$f_\Gamma \leqslant 2D/\log_{\Delta-1}(n).$$

Proof. Clearly, we have that

$$\sum_{(u,v)\in V^2} d(u, v) \leqslant n(n-1)D.$$

Since the degree is constant, we have the equality:

$$1/|E| = 2/n\Delta.$$

Using Lemma 2.1 we are done. □

This simple result shows that "dense" orbital regular graphs have a small edge-forwarding index. Hence we need an upper bound on the diameter. By considering paths starting at 0, we see that $\Gamma_{t,q}$ has diameter D iff every element of $\boldsymbol{F}_q$ can be expressed as a sum of at most D tth powers and some element cannot be expressed as a sum of $D-1$ (or less) tth powers. This is exactly Waring's problem over a finite field [4, 5] which has given rise to an abundant literature on bounds (especially for q, a prime).

In the case where t is fixed and q is large we shall content ourselves with the bound

$$D \leqslant t,$$

which is trivial since $\Gamma_{t,q}$ has t orbitals under the action of S_t, and pairs of vertices in the same orbital are the same distance apart. Then, using Lemma 3.1, we conclude that for t fixed, and large q,

$$f_{\Gamma_{t,q}} \leqslant t(2+\mathrm{o}(1)).$$

More sophisticated bounds would only replace t by $t^{0.5+\varepsilon}$ [5]. As a comparison, for $\Gamma_{2,q}=P_q$ we have, using $\Pi(P_q)=6$, the stronger result

$$f_{P_q} \sim 3.$$

This is the price for not having an accurate estimate of the average distance in $\Gamma_{t,q}$. However, if q is a prime, and $q>t^4$, we know from character sums techniques [4] that

$$D \leqslant 3.$$

In that case, assuming a suitable t (i.e., $t|q-1$ and $t \sim q^{1/4}$) exists, we have the estimate $\log_{\Delta}(n) \sim 4/3$ and

$$f_{\Gamma_{t,q}} \leqslant (9/2 + o(1)).$$

3.3. Covering codes

Using coding theory and covering properties of certain codes it is possible to find orbital regular graphs with an asymptotic merit factor as low as 2.5 [10]. Here we shall give a single example, reserving more general constructions for a forthcoming paper [10]. We shall assume that the reader is already familiar with cyclic codes and covering radius, referring to [7] for basic definitions.

Let C be the binary cyclic code of length $N = 2^m + 1$, m even >2 and having as generator an irreducible polynomial of degree $2m$. This code is named the Zetterberg code; it has minimum distance 5 [9, p. 206] and covering radius 3 [6]. Consider the graph with vertex set the cosets of C in $(GF(2^N), +)$ and edges the pairs $(u+C, v+C)$ where $u+v+C$ has minimum weight 1. Then this graph has diameter the covering radius of C. Since the code has packing radius 2, the number of cosets of weight i is $\binom{N}{i}$ if $i \leqslant 2$. As we know the total number of cosets (2^{2m}), the average distance D_{a} is easy to compute:

$$(2^{2m}-1)D_{\mathrm{a}} = 2^m + 1 + 2(2^{2m-1} + 2^{m-1}) + \tfrac{3}{2}(2^{2m} - 4 - 3 \cdot 2^{2m}).$$

For large m we get $D_{\mathrm{a}} \sim 2.5$. In fact, this graph is $\Gamma_{2^m-1, 2^{2m}}$ (hence orbital regular), as is clear from [7] where the connection between the covering radius problem for certain codes and the Waring problem for finite fields was first noted.

4. Conclusion and open problems

We have introduced the class of orbital regular graphs, whose edge-forwarding index is particularly easy to compute. Interesting examples are provided by a class of Cayley graphs related to the Waring problem over finite fields. Interesting number theoretic questions raised by this research are the following:

- Find two sequences t_m, p_m (p_m, a prime) such that the diameter of Γ_{t_m, q_m} apraoches infinity, but its merit factor remains bounded (it is hoped by less than 3).
- Find better estimates for the average distance in $\Gamma_{t,q}$.

Finally, we have good reasons to believe that many more orbital regular graphs remain to be discovered.

Acknowledgements

We thank Dominique Sotteau for several helpful discussions and for sending us her preprints; an anonymous referee for generalizing the definition of orbital regular;

Wenceslas de La Vega for correcting our faulty definition of the merit factor; and Mark Watkins for helpful advice on the presentation of the material.

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